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# The group theory of oxidation II: Cosets of non-split groups

Arjan Keurentjes<sup>1</sup>*Theoretische Natuurkunde, Vrije Universiteit Brussel,  
Pleinlaan 2, B-1050 Brussels, Belgium*

## Abstract

The oxidation program of reference [4] is extended to cover oxidation of 3-d sigma model theories on a coset  $G/H$ , with  $G$  non-compact (but not necessarily split), and  $H$  the maximal compact subgroup. We recover the matter content, the equations of motion and Bianchi identities from group lattice and Cartan involution. Satake diagrams provide an elegant tool for the computations, the maximal oxidation dimension, and group disintegration chains can be directly read off. We give a complete list of theories that can be recovered from oxidation of a 3 dimensional coset sigma model on  $G/H$ , where  $G$  is a simple non-compact group.

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<sup>1</sup>email address: [Arjan@tena4.vub.ac.be](mailto:Arjan@tena4.vub.ac.be)

# 1 Introduction

Upon dimensional reduction of a physical theory containing gravity to 3 dimensions, and after appropriate dualizations, all degrees of freedom can be represented by scalars. The opposite of dimensional reduction is “oxidation”: Interpreting a theory as a reduction of a higher dimensional one, and reconstructing this higher dimensional theory from the lower dimensional one. In this paper we will be interested in 3 dimensional sigma models on  $G/H$ , where  $G$  is a non-compact group and  $H$  its maximal compact subgroup, and in the theories that can be oxidized from them.

In [1] a list of 3 dimensional theories based on coset spaces that are oxidizable appeared, together with their 4 dimensional counterparts. In [2] an analysis of oxidation from 3 dimensional theories on coset spaces of the form  $G/H$ , with  $G$  split (maximally non-compact) was presented; many such theories can actually be oxidized to more than 4 dimensions. An interesting aspect of oxidation is that it is not unique; there may be different higher dimensional theories leading to the same lower dimensional theory. In [3] and [4] such branches are analyzed from different points of view. The paper [3] builds on developments from [5] and explores a suggestive link with Del Pezzo surfaces [6]. In [4] we connected some old ideas [7, 8] to a systematic recipe for dimensional reduction as developed in [9, 10, 2]. Via a recasting of the sigma model equations, we were able to do an exhaustive analysis of theories with split groups, using mainly Lie group theory, and we rederived (and extended) the results of [2] from this unified perspective.

It seems that so far there was no systematic understanding for theories oxidized from cosets that are not based on split groups. It is hard to deal with these with the methods of [3], as it is not (yet) clear how to connect the Del Pezzo/gravity correspondence to groups that are not subgroups of  $E_{8(8)}$ . The methods of [4] however can be straightforwardly extended to this more general case, and this will be the topic of this paper.

Most of the theories we find appear to have been described already in the literature. Our analysis is systematic, and includes for example the bosonic sectors of the  $N = 2$  theories related by the **r**- and **c**-maps [11, 12, 13], insofar as these give rise to symmetric spaces (the other spaces result in higher dimensional theories with supergravity as a subsector, and therefore must have subspaces that are symmetric spaces).

In section 2 we will revisit and extend the methods of [4] to cover cosets from generic non-compact groups. Apart from the introduction of some extra elements from group theory, the analysis is very similar to the one for the split groups. Many of the results from [4] carry over straightforwardly.

In section 3 we discuss the diagrammatic language of Satake diagrams. These are helpful with the analysis of the cosets as they encode all the relevant information about the non-compact group. They also capture the full process of “group disintegration”; in particular the maximal oxidation dimension, and the non-compact forms appearing in the oxidation

procedure can be directly read off.

In section 4 we analyze theories oxidized from non-compact cosets, including  $G/H$  with  $G$  a group over the complex numbers, and all cosets based on non-compact simple groups.

In a final section we summarize, and highlight some significant points.

Almost all our conventions regarding Lie groups can be found in the appendices of [4]. Necessary additional material will be introduced in section 2.

## 2 General theory of oxidation

Our philosophy is mainly based on ideas from [7] (where they were discussed in the context of supergravities) and can be found in section 2 of [4]. These ideas are quite general, and will also be the basis for our analysis here. We briefly repeat the main points, and refer to section 2 of [4], and references therein for more details.

The starting point is a coset sigma model in 3 dimensions on  $G/H$ , with  $G$  a non-compact group, and  $H$  its maximal compact subgroup (one can include the pathological case where  $G$  is compact; then  $G = H$ , the coset is just a point and the sigma model is empty). The dimensional reduction of general relativity from  $D$ -dimensions to 3 dimensions results in a sigma model on  $SL(D-2, \mathbb{R})/SO(D-2)$  coset [10]. Interpreting the  $G/H$ -model in 3 dimensions as resulting from some higher dimensional theory, the gravity sector can be recovered by studying the embeddings of  $SL(D-2, \mathbb{R}) \subset G$ . Decomposing  $G$  into  $SL(D-2, \mathbb{R})$  irreducible representations (irreps), we find at least the adjoint irrep of  $SL(D-2, \mathbb{R})$ , to be interpreted as the graviton. We demand exactly one graviton, and want to interpret the remaining irreps as massless fields (forms). This translates into a constraint on the possible embeddings of  $SL(D-2, \mathbb{R})$  in  $G$ : They have to be *index 1* embeddings [4]. Equivalently, the roots of  $SL(D-2, \mathbb{R})$  can be chosen such that they coincide with *long* roots of  $G$ .

As the adjoint irrep of  $G$  is a self-conjugate representation, we must find self conjugate representations of  $SL(D-2, \mathbb{R})$  in the composition. These come in two kinds: Self-conjugate irreps, and pairs of mutually conjugate irreps. The self-conjugate irreps are, due to the restriction to level 1 subgroups, the adjoint (representing the graviton), singlets (scalars), and self-conjugate  $(D-2)/2$  tensors (if  $D$  is even; we will discuss these momentarily). The pairs of mutually conjugate irreps, are always combination of an  $n$ -form and a  $(D-2-n)$  form; these represent a form field and its dual. In [4], we associated an equation to every  $SL(D-2, \mathbb{R})$ -irrep: the adjoint irrep is linked to the Einstein equation, every  $n$ -form gives a Bianchi identity for a form field  $F_{(n+1)}$ , the  $(D-2-n)$ -form gives the equation of motion for the same form. There is no fundamental distinction between the  $n$ - and the  $(D-2-n)$ -form; this is how the possibility to dualize fields is built in in our theory. A self-conjugate tensor irrep gives rise to either a self-dual tensor for  $D-2 = 4k$ , or

“pseudo-selfduality” (with the imaginary unit  $i$  occurring in the duality equation) for  $D = 4k$ ; in both cases one has only half the number of equations, and degrees of freedom.

The centralizer of  $SL(D-2, \mathbb{R})$  in  $G$  acts as a symmetry group on the theory. We call this group the “U-duality group”, and denote it by  $U_D$ .

Recovering the level 1 embeddings of  $SL(D-2, \mathbb{R})$  in  $G$  is quite easy if  $G$  is split. Then one can pick a basis for the Lie algebra of  $G$ , consisting of Cartan generators  $H_i$ , and ladder operators  $E_{\pm\alpha}$ , such that  $G$  is generated by linear combinations of these generators with *real* coefficients. Picking a sublattice of  $G$ , the corresponding generators generate a group that is also split; picking a  $A_{D-3}$  sublattice of long roots, the corresponding generators generate  $SL(D-2, \mathbb{R})$  (and not another real form). This last statement is not true for generic non-compact groups, and requires us to introduce some more technology.

## 2.1 Non-compact groups

A notion that is central to the study of non-compact real forms of semi-simple Lie-groups is that of a Cartan involution [14]. From the Cartan involution the non-compact real form of the group can be easily reconstructed.

An involutive automorphism  $\theta$  is called a Cartan involution if  $-\langle X, \theta Y \rangle$  is strictly positive definite for all  $X, Y$ . An involution has eigenvalues  $\pm 1$ , and the realization of the involution can be chosen such that the Cartan subalgebra is closed under the involution.

Under the action of  $\theta$  on the Cartan subalgebra generators  $\alpha = \alpha_i H_i$ , the root space  $\mathcal{H}$  decomposes into two orthogonal complements. The space spanned by eigenvectors of  $\theta$  with eigenvalue 1 we call  $\mathcal{H}_+$ , its complement  $\mathcal{H}_-$ . The lattice in the subspace  $\mathcal{H}^+$  containing the invariant roots  $\theta(\alpha) = \alpha$  is the lattice of a compact group  $G_c$ . We denote the set of invariant roots by  $\Delta_c$ . Similarly, we use  $\Delta_c^+$  for  $\Delta^+ \cap \Delta_c$ . The generators of  $G_c$  are  $iH_j$ ,  $E_\alpha - E_{-\alpha}$ , and  $i(E_\alpha + E_{-\alpha})$ . We emphasize however that  $G_c$  is *not* a maximal compact subgroup; it is not even a maximal regular compact subgroup. It will however play a crucial role in our construction of the sigma model.

The regular subgroup commuting with  $G_c$  will be called  $G_s$ . The roots of  $G_s$  obey  $\theta(\alpha) = -\alpha$ , and the set of roots of  $G_s$  will be denoted by  $\Delta_s$ . The generators of  $G_s$  are of the form  $H_j$ ,  $E_\alpha - E_{-\alpha}$ , and  $E_\alpha + E_{-\alpha}$ . The group  $G_s$  is a regular split group; it need however not be the maximal regular split group (a counterexample is found for  $SO(p, q)$  with  $p$  or  $q$  odd) nor does it imply that it is an index 1 subgroup (a counterexample is found for  $F_{4(-20)}$ ). We recover the index 1  $SL(D-2, \mathbb{R})$ -subgroups required for oxidation as subgroups of  $G_s$ ; nevertheless, the group  $G_s$  is not as important to our analysis as  $G_c$ .

The remaining roots mix under the Cartan involution. We define the image of the ladder operator  $E_\alpha$  to be  $C_\alpha E_{\theta(\alpha)}$ , with  $C_\alpha = \pm 1$ , where the plus- or minus sign should be chosen consistently with a number of conditions, that we will describe now. First of all, if  $\alpha \in \Delta_c$ ,

$C_\alpha = 1$ . If  $\alpha \in \Delta_s$ , we choose  $C_\alpha = -1$ . If  $\alpha \notin (\Delta_c + \Delta_s)$ , closure of the algebra combined with the fact that  $\theta$  is an automorphism leads to the requirement

$$C_{\alpha+\beta} N_{\alpha,\beta} = C_\alpha C_\beta N_{\theta(\alpha),\theta(\beta)}$$

This may still leave some signs unfixed, but this is not important for what follows.

To specify a non-compact real form, one starts from the compact real form, and a Cartan involution. With the above considerations, it suffices to specify the action on the root space (and fix a convention for the  $C_\alpha$ ). We then divide the generators of the compact form in a set invariant under the Cartan involution, and a set that has eigenvalue  $-1$ . As generators for the corresponding real form, we take the invariant generators, and add to these the non-invariant generators multiplied by the imaginary unit  $i$ . The invariant generators generate a compact subgroup; the non-invariant generators turn into “non-compact” generators by multiplication by  $i$ . Notice that in the extreme cases that  $G$  is compact, or  $G$  is split, one immediately recovers the generators specified previously.

Two important characteristics of non-compact groups are the  $\mathbb{R}$ -rank (real rank) and the character.

The  $\mathbb{R}$ -rank can be defined as follows: A Cartan subalgebra<sup>2</sup>  $\mathcal{H}$  is a maximal Abelian subalgebra with  $\text{ad}(\mathcal{H})$  completely reducible. The Cartan subalgebra generates an Abelian group called a *torus* (Beware: in this context a torus has the topology  $(S^1)^m \times \mathbb{R}^n$  for some  $m, n \geq 0$ , and hence is only a torus in the usual sense if  $n = 0$ ). A torus is  $\mathbb{R}$ -split if it is diagonalizable over  $\mathbb{R}$  (and hence one which has  $m = 0$  instead of  $n!$ ). The  $\mathbb{R}$ -rank is defined as the dimension of a maximal  $\mathbb{R}$ -split torus. It equals the multiplicity of the eigenvalue  $-1$  of the restriction of  $\theta$  to  $\mathcal{H}$ , and therefore  $\dim(\mathcal{H}_-)$ . The  $\mathbb{R}$ -rank is maximal for the split form of a group, and then coincides with the rank. Its minimal value is zero, for compact forms.

The character is more easily defined. Let  $G$  be a non-compact group, and  $H$  its maximal compact subgroup. Then there are  $d_- = \dim(H)$  compact generators, while the complement consists of  $d_+ = \dim(G) - \dim(H)$  “non-compact generators”. The character  $\sigma$  is defined as  $\sigma = d_+ - d_-$ . The maximal value of the character coincides with the rank, and is obtained for the split groups (and then automatically equals the  $\mathbb{R}$ -rank). The minimum value for the character is obtained if  $d_+ = 0$ , hence for compact  $G$ , when  $\sigma = -\dim(G)$ .

Suppose we start with a coset sigma model on  $G/H$  in 3 dimensions, with  $G$  a group with character  $\sigma_3$  and  $\mathbb{R}$ -rank  $r_3$ . If this theory is oxidizable to  $D$  dimensions, one finds a U-duality group with character  $\sigma_D$  and  $\mathbb{R}$ -rank  $r_D$  are given by [7]

$$\sigma_D = \sigma_3 - (D - 3); \quad r_D = r_3 - (D - 3). \quad (1)$$

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<sup>2</sup>We use the same symbol for the root space and the Cartan subalgebra, as the two can be identified.

The statement on the character takes into account the possible presence of compact factors. Such factors are not manifest in the sigma model (they drop out after division by the maximal compact subgroup), but do impose important restrictions on the matter sectors (that have to organize in representations of these compact factors). Omitting such compact factors leads to a break-up of the pattern. With this proviso, the above statements are actually easy to prove.

Consider the algebra elements  $H_i$ ,  $E_{\pm\alpha}$ , and the Cartan involution acting on the root space. The oxidation recipe demands a level 1  $A_{D-3}$  sublattice of the root lattice; because  $SL(D-2)$  is split, this lattice is contained in a subspace on which the Cartan involution  $\theta$  acts as  $-1$ . Hence on the complementary subspace, the Cartan involution acts  $\text{diag}(1, \dots, 1, -1, \dots, -1)$ , where the number of minus signs is  $r_D = r_3 - (D-3)$ .

Regarding the character of the groups, let us decompose the root lattice. The root lattice of  $SL(D-2)$  is a sublattice, the complementary subspace has the lattice of  $U_D$ , and there are a number of roots that have components in both subspaces. Let us start with the latter. Consider a root  $\alpha$  of  $G$ , where  $\alpha$  has components in the direction of the  $SL(D-2)$  lattice. The Cartan involution  $\theta$  acts as  $-1$  on the subspace containing the  $SL(D-2)$ , and we can form the generators  $E_\alpha \pm C_\alpha E_{\theta(\alpha)}$ . The combination with the plus sign is invariant under the Cartan involution, and hence corresponds to a compact generator (upon combination with its hermitian conjugate), while the combination with the minus sign gives a non-compact generator. Important is that in this way, we find that roots with components in both subspaces give rise to equal numbers of compact and non-compact generators, and hence do not contribute to the character. Hence:

$$\sigma_3 = \sigma(G) = \sigma(U_D) + \sigma(SL(D-2)) = \sigma_D + (D-3). \quad (2)$$

Note that regularity of the subalgebra's plays an important role in the proof.

## 2.2 Sigma models on non-compact cosets

As in [4], we base our discussion on the following form of the sigma model action ( $\mathcal{V} \in G$ )

$$L_{G/H} = -e \, \text{tr} \left( (\partial\mathcal{V})\mathcal{V}^{-1} \frac{1}{2}(1+T)(\partial\mathcal{V})\mathcal{V}^{-1} \right). \quad (3)$$

The operator  $T$  acts on algebra elements  $A$  as  $T(A) = -\theta(A)$ ; because  $\theta$  is an involution  $\frac{1}{2}(1+T)$  is a projection operator.

The form  $(d\mathcal{V})\mathcal{V}^{-1}$  can be expanded in generators as follows:

$$(d\mathcal{V})\mathcal{V}^{-1} = \frac{1}{2} \sum_{i=1}^r d\phi^i H^i + \sum_{\alpha \in \Delta_{nc}^+} e^{\frac{1}{2}\langle\alpha, \phi\rangle} F_{(1)\alpha} E_\alpha. \quad (4)$$

This expression assumes a particular gauge, that is implicit in the choice of symbols. The constant  $r$  denotes the  $\mathbb{R}$ -rank; the  $H_i$  form a basis for a maximal  $\mathbb{R}$  split torus. The symbol  $\Delta_{nc}^+$  denotes  $\Delta^+ - \Delta_c^+$ , the set of positive roots of  $G$  that are not roots of  $G_c$ . That one can choose this gauge follows from the Iwasawa decomposition<sup>3</sup>. Equation (4) is almost identical to equation (25) in [4]; for split  $G$  it is identical, as then  $r$  coincides with the rank, and  $\Delta_c^+ = \emptyset$ .

Another modification is that  $\phi$  only has components for the directions corresponding to the  $\mathbb{R}$ -split torus. Because there are fewer dilatons, the inner products  $\langle \alpha, \phi \rangle$  are defined by setting those components of  $\phi$  in the direction transverse to the  $\mathbb{R}$ -split torus to zero. An amusing observation is that the dilatons can be identified with elements of the Cartan subalgebra; elements of the Cartan subalgebra with negative norm (compact generators) would lead to kinetic terms with wrong signs, but are projected out by the denominator compact subgroup.

All formula's of the sections 4.1 and 4.2 of [4] can be copied to the more general case, by setting  $r$  to be the  $\mathbb{R}$ -rank, replacing  $\Delta^+$  by  $\Delta_{nc}^+$ , and remembering to modify the inner products involving  $\phi$  by setting the appropriate components of  $\phi$  to zero. These modifications do not change any derivation from [4].

Hence we find the Bianchi identities:

$$dF_{(1)\gamma} = \frac{1}{2} \sum_{*} N_{\alpha,\beta} F_{(1)\alpha} \wedge F_{(1)\beta} \quad * = \left\{ \begin{array}{l} \alpha, \beta, \gamma \in \Delta_{nc}^+ \\ \alpha + \beta = \gamma \end{array} \right. ; \quad (5)$$

and the equations of motion:

$$dF_{(D-1)-\gamma} = \sum_{*} N_{\alpha,-\beta} F_{(1)\alpha} \wedge F_{(D-1)-\beta} \quad * = \left\{ \begin{array}{l} \alpha, \beta, \gamma \in \Delta_{nc}^+ \\ \alpha - \beta = -\gamma \end{array} \right. , \quad (6)$$

where we have defined

$$F_{(D-1)-\gamma} \equiv e^{\langle \gamma, \phi \rangle} * F_{(1)\gamma} = dA_{(D-2)-\gamma} - \sum_{\beta-\alpha=-\gamma} N_{\beta,-\alpha} F_{(1)\beta} \wedge A_{(D-2)-\alpha}. \quad (7)$$

It seems there is no longer a one on one relation between algebra generators and equations, as in [4]. This relationship is recovered when supplementing the equations of motion and Bianchi identities with algebraic equations for the missing generators:

$$\phi_i = 0 \text{ for } i > r \quad F_{(n)\alpha} = 0 \text{ for } \alpha \in \Delta_c. \quad (8)$$

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<sup>3</sup>This way of parameterizing the sigma model leads to the identity

$$\dim(G) - \dim(H) = \frac{1}{2}(\dim(G) + r(G)) - \frac{1}{2}(\dim(G_c) + r(G_c)).$$

The ranks of  $G, G_c$  are denoted by  $r(G), r(G_c)$ . Left and right hand side express the number of scalars, the left hand side from the abstract definition, the right hand side from the gauge choice; the two terms in brackets give the number of positive roots plus the dimension of the Cartan subalgebra of  $G$ , and  $G_c$ .

These equations are a consequence of the Iwasawa decomposition, reflected in the choice of gauge. For  $G$  compact, all equations are algebraic, and the model empty.

Also the addition of matter proceeds analogously to [4]. The matter Bianchi identity becomes

$$dF_{(n)\lambda'} = \sum_{*} N_{\alpha,\lambda} F_{(1)\alpha} \wedge F_{(n)\lambda} \quad * = \begin{cases} \lambda, \lambda' \in \Lambda \\ \alpha \in \Delta_{nc}^{+} \\ \alpha + \lambda = \lambda' \end{cases} ; \quad (9)$$

and the equation of motion becomes

$$dF_{(D-n)-\lambda'} = \sum_{*} N_{\alpha,-\lambda} F_{(1)\alpha} \wedge F_{(D-n)-\lambda} \quad * = \begin{cases} \lambda, \lambda' \in \Lambda \\ \alpha \in \Delta_{nc}^{+} \\ \alpha - \lambda = -\lambda' \end{cases} . \quad (10)$$

Again the set of weights  $\Lambda$  belongs to a certain  $G$  representation, and  $\bar{\Lambda}$  to the conjugate representation. We remind the reader of the possibility that form and dual form transform in a self-conjugate representation, which must be realized in theories with self-dual forms. Then the equation of motion (10) and Bianchi identity (9) are the same equation, and we can consistently impose self-duality.

Again we have one Bianchi identity, and one equation of motion, labelled by  $\lambda$  respectively  $-\lambda$ . For self-dual representations constraint equation and Bianchi identity imply each other, but as  $\lambda$  and  $-\lambda$  belong to the same representation we precisely get as many equations as weights.

The equation of motion for the dilatonic scalars becomes

$$2d(*d\phi^i) = \sum_{\alpha \in \Delta_{nc}^{+}} \alpha^i F_{(D-1)-\alpha} \wedge F_{(1)\alpha} + \sum_{\lambda \in \Lambda} \lambda^i F_{(D-n)-\lambda} \wedge F_{(n)\lambda}. \quad (11)$$

In absence of matter, the second sum drops out.

## 2.3 Oxidation

The equations of the previous section are almost identical to the ones appearing in [4]. It should come as no surprise that also the oxidation recipe is hardly modified.

To be precise, there is only one small modification in the oxidation recipe for the axions. The assignment of forms to the antisymmetric tensor representations of  $SL(D-2)$  proceeds as in [4]. The difference is in the singlets of  $SL(D-2)$ ; these correspond to the the group  $U_D$ . In [4] we made a positive root decomposition of the semi-simple part of  $U_D$ , while choosing positive directions for the Abelian factors. Here we do the same, but on top of that we have to identify the roots of the group  $G_c \subset U_D$ . We call this set of



roots  $\Delta_c$ . We again associate 1-forms  $F_{(1)\alpha}$  to positive roots, and  $D - 1$ -forms  $F_{(D-1)-\alpha}$  to the negative roots. If  $\alpha$  is a root of  $\Delta_c$ , then so is  $-\alpha$ , and we set

$$F_{(k)\pm\alpha} = 0, \quad k = 1, D - 1. \quad (12)$$

The remaining equations are again given by

$$dF_{(n)\alpha'} = \frac{1}{2} \sum_* \eta_{l,\beta;m,\gamma} N_{\beta,\gamma} F_{(l)\beta'} \wedge F_{(m)\gamma'} \quad * = \left\{ \begin{array}{l} l + m = n + 1 \\ \alpha' + \beta' = \gamma' \end{array} \right. . \quad (13)$$

Computation of the sign factors  $\eta_{l,\beta;m,\gamma}$  proceeds as explained in [4]. We repeat once more that the only difference is in exponential prefactors not manifest in (13); we have

$$F_{(D-n)-\alpha} \equiv e^{\langle \alpha, \phi \rangle} * F_{(n)\alpha} \quad (14)$$

with  $\phi$  including some components set to zero from the start.

The dilaton equation is always (11), where  $i$  is restricted to run from 1 to the  $\mathbb{R}$ -rank  $r$ : Components transverse to the  $\mathbb{R}$ -split torus play no role.

It should be clear that our comparison of this oxidation recipe to dimensional reduction, in section 4.5 of [4] requires no repetition or modification.

### 3 Satake diagrams

So far we have used the abstract description of non-compact groups and their algebra's. For specific computations, we have to specify the (complexified) algebra, and the Cartan involution. These can be encoded in so-called Satake diagrams [14].

#### 3.1 Non-compact groups from diagrams

A Satake diagram is a Dynkin diagram, with additional “decoration” (Satake diagrams for all simple real forms can be found in the figures 3-11). In a traditional Dynkin diagram all nodes are of the same colour, in a Satake diagram we use black (solid) dots for some nodes, and white (open) dots for others. On top of that, there is the possibility of connecting certain nodes by an arrow. The purpose of these decorations is to encode the Cartan involution  $\theta$ . It can be reconstructed as follows.

A black node stands for a simple root that is invariant under the involution  $\theta$  (and therefore, it will be a root of  $G_c$ ). A white node corresponds to a simple root that is not invariant under  $\theta$ . White nodes can be connected by arrows; if the nodes corresponding to  $\alpha$  and  $\beta$  are connected, this signifies that  $\alpha - \alpha^\theta = \beta - \beta^\theta$ . As it stands, this means

that  $\alpha$  and  $\beta$  have the same component on the subspace of eigenvalue  $-1$  under  $\theta$ . Some reshuffling teaches us that  $\alpha - \beta$  belongs to the invariant subspace. Hence a basis for the invariant subspace is formed by the roots corresponding to black nodes, together with the differences of the roots connected by arrows. On the orthogonal complement  $\theta$  has eigenvalue  $-1$ . This completes the specification of  $\theta$ , and hence of the real form.

Note the two extremes: we can have diagrams of black dots only, meaning  $\theta = 1$ , and hence the real form is the compact one; a diagram of white dots only, and no arrows indicates  $\theta = -1$ , and hence the real form is the split form.

### 3.2 Group disintegration

One might wonder if Satake diagrams have a role to play in “group disintegration”, the chain of subgroups of  $G$  one finds in oxidation. This is indeed the case.

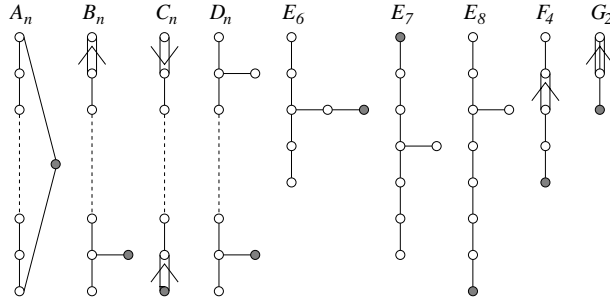


Figure 1: Extended Dynkin diagrams; the marked node is the extended one. How to produce extended Satake diagrams from these is explained in the text.

Let us consider “Extended Satake diagrams”; analogously to Satake diagrams, we define these as Extended Dynkin diagrams with decoration. We copy the decoration of all nodes from the Satake diagram; only the decoration of the extended node has to be specified. This is easy upon using the fact that the Satake diagram specifies  $\theta$  completely. The extended root is linearly dependent on the other roots; the corresponding node *can* only be black if all the other nodes are black, and in that case, it *must* be black. There can be no arrows pointing to the extended node; its partner would have to be contained in the Satake diagram, contradicting that the Satake diagram completely encodes the involution.

We know that the Extended Dynkin diagram encodes the regular subgroups of  $G$ ; as always we want to decompose in an  $SL$  chain and a complementary subgroup. It is an old observation [8, 2] that group disintegration has to start at the end of the Dynkin diagram where the affine vertex attaches; in [4] we argued that this is so because one can find the U-duality group by taking the extended Dynkin diagram, starting an  $SL$  chain

at the extended node, and erase the appropriate nodes to disconnect it from the rest of the diagram.

Exactly the same recipe works for Extended Satake diagrams. Provided we are not in the “all black nodes” case (corresponding to the compact form, implying a trivial coset, and hence no oxidation), the extended node is white. As  $SL$  groups are split, they are composed entirely of white nodes, hence we look for chains of these. The nodes that we erase are not allowed to be black; black nodes correspond to the invariant subspace, which cannot become lower dimensional; the  $SL$  roots are entirely in the orthogonal complement. Lastly, the remaining diagram, complementary to the  $SL$  chain must make sense as a Satake diagram. Sticking to these rules, the Satake diagram complementary to the  $SL$  chain gives the semi-simple part of the U-duality group. We again have to complete by adding Abelian factors if the final (not extended) diagram has less nodes than the (not extended) diagram we started with. Erased nodes were always white, but there was the possibility of some nodes being connected by arrows. A direct computation reveals that a pair of erased nodes connected by an arrow gives a  $U(1)$ -factor, while a surplus of erased white nodes without arrows gives  $\mathbb{R}$ -factors.

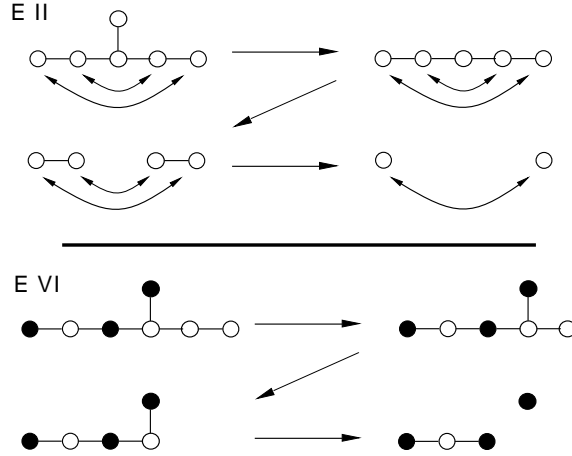


Figure 2: Examples of group disintegration for Satake diagrams

We depicted 2 examples in figure 2. The first example is  $E_{6(2)}$ , oxidizing in steps to 6 dimensions. The extended node connects to the branch of the diagram not decorated by arrows. The group disintegration proceeds along this branch leading to the diagrams of  $SU(3, 3)$  and  $SL(3, \mathbb{C})$ , which are the groups following from an explicit computation [13]. In the last step we loose two nodes connected by an arrow; the remaining diagram is  $SL(2, \mathbb{C}) \cong SO(3, 1)$ , and as explained in the above, a  $U(1)$ -factor should be added.

The second oxidation chain depicts the decompactification of the bosonic sector of  $N = 6$  supergravity. Subsequently we find the diagrams of  $SO^*(12)$ ,  $SU^*(6)$  and  $SU(2) \times SU^*(4)$ .

Note that in the last step we find a disconnected black node, giving an  $SU(2)/SU(2)$  factor in the U-duality group. This 6 dimensional chiral supergravity was considered in [15]. The equations of motion and Bianchi identities appear not to have been written down, but can be straightforwardly constructed from our formalism (for the bosonic sector).

## 4 Results for various cosets

We now apply the formalism of the previous section to sigma models based on cosets of  $G/H$ , for two important classes of non-compact  $G$ . The first class are simple Lie groups over the complex numbers, the second consists of real forms of simple Lie groups.

### 4.1 Groups over the complex numbers

Consider the complexification of a compact simple Lie algebra that generates a group  $H$ . The complexified algebra generates a group that we will denote by  $H^{\mathbb{C}}$ . The maximal compact subgroup is the compact form of  $H$ . The Satake diagrams for these consist of two copies of the Dynkin diagram of  $H$ , with *open* nodes, where the nodes in the two copies are connected pairwise by arrows. Every symmetric combination of generators of the two copies gives rise to a non-compact generator, while every anti-symmetric combination results in a compact generator. Therefore, the character  $\sigma$  is always zero.

As is easily seen, the invariant subspace contains no roots, nor does its complement. The group  $G_s \times G_c = (\mathbb{R} \times U(1))^r$  (which is a torus), with  $r$  the rank of the algebra  $H$  (and the  $\mathbb{R}$ -rank of  $H^{\mathbb{C}}$ ). Therefore, a sigma model in 3 dimensions on  $H^{\mathbb{C}}/H$  can never be oxidized. Note that it is certainly possible for  $H^{\mathbb{C}}$  to contain  $SL$  subgroups, but these have to be index 2 or higher, and therefore do not meet our constraints. A well known example is  $SL(2, \mathbb{C})$ , the double cover of the 4 dimensional Lorentz group  $SO(3, 1)$ , which is known to be not oxidizable.

Next a systematic survey of all non-compact real forms of simple Lie algebra's is presented. Many theories that we reconstruct from these are known as bosonic sectors from the context of supergravity theories. The non-compact real forms have been classified in chains, labelled by a capital and a roman number [14]. The capital coincides with the one assigned to the (complexified) algebra. For each non-compact real form we list its Satake diagram, its maximal compact subgroup  $H$ , its character  $\sigma$ , its  $\mathbb{R}$ -rank  $r$ , and the groups  $G_s$ , and  $G_c$ . In the cases where the 3 dimensional sigma model is oxidizable, we present the  $U$ -duality groups in the oxidation chain. To recover the matter content of the theory, we refer the reader to the decompositions in appendix B of [4]; these can be applied directly to the non-split non-compact forms, upon noting that one only has to replace the split  $U$ -duality groups centralizing  $SL(D-2, \mathbb{R})$  by the non-compact forms in

the tables in this paper. Note that the oxidation chains for the split forms are always longer than those of the non-split forms, so the table is truncated for higher dimensions. For completeness we mention the split non-compact forms, but we will omit most details about them, as these can be found in [2, 4] and references therein.

## 4.2 $A_n$

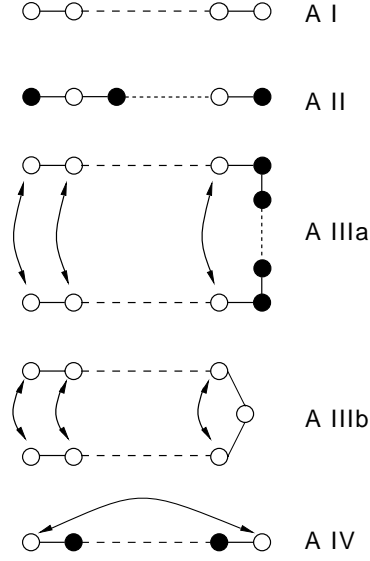


Figure 3: Satake diagrams for non-compact forms of the  $A_n$  groups

### 4.2.1 $AI$ : $SL(n+1, \mathbb{R})$

This is the split non-compact form the  $A_n$  algebra. These oxidize to general relativity in  $n + 3$  dimensions [10], for the  $SL(2, \mathbb{R})/SO(2)$  case this has been known for a long time. The maximal compact subgroup, character, and  $\mathbb{R}$ -rank are

$$H = SO(n+1); \quad \sigma = n; \quad r = n.$$

For split forms  $G = G_s$  and  $G_c$  is the trivial group:

$$G_s \times G_c = SL(n+1, \mathbb{R}) \times \{e\}$$

For details we refer to [10].

#### 4.2.2 A II: $SU^*(n+1)$

For this non-compact form we require  $n > 1$  ( $SU^*(2) = SU(2)$ , the compact form), and  $n$  odd. The maximal compact subgroup, character and  $\mathbb{R}$ -rank are

$$H = Sp(\frac{1}{2}(n+1)); \quad \sigma = -n-2; \quad r = \frac{1}{2}(n-1)$$

The subgroups  $G_s$  and  $G_c$  are found to be:

$$G_s \times G_c = \mathbb{R}^{\frac{1}{2}(n-1)} \times SU(2)^{\frac{1}{2}(n+1)}$$

As is obvious from  $G_s$ , sigma models based on these cosets cannot be oxidized, due to the absence of a level 1  $SL(n)$ -group. This was already noted in [1].

#### 4.2.3 A III, A IV: $SU(n+1-p, p)$

As obviously  $SU(a, b)$  is isomorphic to  $SU(b, a)$  we restrict to  $p \leq \frac{1}{2}(n+1)$ . In the list of Satake diagrams (fig 3) we have distinguished A IIIa with  $1 < p < \frac{1}{2}(n+1)$ , and A IIIb where  $p = \frac{1}{2}(n+1)$ . The series A IV refer to the case  $p = 1$ . Note that  $SU(1, 1) \cong SL(2, \mathbb{R})$ .

The maximal compact subgroup, character and  $\mathbb{R}$ -rank are given by

$$H = S(U(n+1-p) \times U(p)); \quad \sigma = 1 - (n+1-2p)^2; \quad r = p$$

In computing  $G_s$  and  $G_c$  we have to distinguish between A IIIa and A IV on the one hand, and A IIIb on the other hand. If  $n+1 \neq 2p$  (A IIIa and A IV)

$$G_s \times G_c = SL(2, \mathbb{R})^p \times SU(n+1-2p) \times U(1)^p$$

while for  $n+1 = 2p$  (A IIIb)

$$G_s \times G_c = SL(2, \mathbb{R})^p \times U(1)^{p-1}$$

From  $G_s$  we see that these theories always oxidize to 4 dimensions (with  $SU(1, 1) \cong SL(2, \mathbb{R})$  fitting perfectly in the pattern). The relevant groups for the oxidation are:

$D$	$G/H$
4	$SU(n-p, p-1) \times U(1) / S(U(n-p) \times U(p-1)) \times U(1)$
3	$SU(n+1-p, p) / S(U(n+1-p) \times U(p))$

The 4 dimensional theory contains general relativity, coupled to  $n-1$  vectors, and a sigma model. Note that the  $U(1)$  factor cancels out of the sigma model, though the vectors carry charges under the  $U(1)$ . For  $p = 1$  the 4-d sigma model is empty.

For  $p = 2$ , and  $SU(2, 1)$  the symmetric spaces of the 3-d sigma model are quaternionic and relevant in the context of  $N = 2$  supergravity. For these the process of oxidation is the inverse of the **c**-map [12].

### 4.3 $B_n$

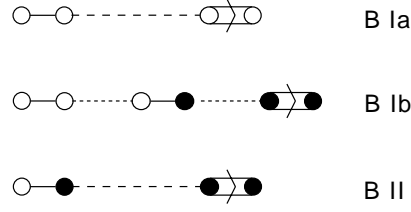


Figure 4: Satake diagrams for non-compact forms of the  $B_n$  groups

#### 4.3.1 $BI, BII$ : $SO(2n+1-p, p)$

These models are very similar to those in based on  $DI, DII$  (which we will discuss in subsection 4.5.1), and in the existing literature are often treated simultaneously. As obviously  $SO(a, b) \cong SO(b, a)$ , we restrict to  $p \leq n$ . In the list of Satake diagrams we have distinguished between  $B Ia$ , with  $p = n$  (the split form), and  $B Ib$  where  $1 < p < n$ . The case  $B II$  refers to  $p = 1$ .

The maximal compact subgroup, character and  $\mathbb{R}$ -rank are given by

$$H = SO(2n+1-p) \times SO(p); \quad \sigma = n - 2(n-p)(n-p+1); \quad r = p$$

For the split case  $B Ia$  one has of course

$$G_s \times G_c = SO(n+1, n) \times \{e\},$$

while for the other cases

$$G_s \times G_c = SO(p, p) \times SO(2n-2p+1)$$

The table of groups appearing in the oxidation chain is

$D$	$G/H$
$p+2$	$SO(2n-2p+1) \times \mathbb{R} / SO(2n-2p+1)$
$\vdots$	$\vdots$
$d+2$	$SO(2n-p-d+1, p-d) \times \mathbb{R} / SO(2n-p-d+1) \times SO(p-d)$
$\vdots$	$\vdots$
6	$SO(2n-p-3, p-4) \times \mathbb{R} / SO(2n-p-3) \times SO(p-4)$
5	$SO(2n-p-2, p-3) \times \mathbb{R} / SO(2n-p-2) \times SO(p-3)$
4	$SO(2n-p-1, p-2) \times SL(2, \mathbb{R}) / SO(2n-p-1) \times SO(p-2) \times SO(2)$
3	$SO(2n-p+1, p) / SO(2n-p+1) \times SO(p)$

Furthermore, for  $p > 3$  there is an alternative decomposition leading to

$D$	$G/H$
6	$SO(2n-p-2, p-3)/SO(2n-p-2) \times SO(p-3)$

An exceptional case is  $n = p = 1$ , where one has  $SO(2, 1) \cong SL(2, \mathbb{R})$ . For  $SO(2q, 1)$ ,  $q \neq 1$ , there is no oxidation possible.

In the other cases the theory can be oxidized to  $p+2$  dimensions, where one finds an Einstein-dilaton type gravity, coupled to an antisymmetric tensor, and  $2n-2p+1$  vectors. The latter form the vector representation of  $SO(2n-2p+1)$ , though this compact factor cancels from the sigma model.

As for the split forms [4] there is (for  $p > 3$ ) a separate branch in 6 dimensions, leading to a theory with general relativity, a sigma model, and (anti-)self dual tensors. The number of self dual and anti-self dual tensors is not equal, and the theory is chiral, even in absence of fermions.

As in the split case [4] one can analyze the explicit theory in  $p+2$  dimensions, to show that the Bianchi identity for the 2-form takes the form

$$dF_{(3)} = \frac{1}{2} \sum_{i=1}^{2n-2p+1} F_{(2)i} \wedge F_{(2)i} \quad (15)$$

which is reminiscent of the identity  $dH = \frac{1}{2}\text{Tr}(F \wedge F) - \frac{1}{2}\text{Tr}(R \wedge R)$  from string theory. Indeed, these models follow from the general considerations of [16] for string theories. The models based on  $SO(8, 2q+1)$  have supersymmetric extensions to theories with 16 supersymmetries; in 10 dimensions, these give type I supergravity coupled to an odd number of Yang-Mills multiplets [17]. Though a priori of relevance to string theory, there appear to be very few realizations of these models from superstrings [18].

Other interesting theories are the ones based on  $SO(4, 2q+1)$  (note that  $SO(4, 1) \cong Sp(1, 1)$ ) that give rise to quaternionic manifolds in 3 dimensions, and are important in the context of theories with 8 supersymmetries [13]. The 2 different 6 dimensional branches correspond to exchanging vector multiplets with tensor multiplets. A string theory realization can be constructed by compactifying a heterotic string theory on  $K3$ , with a number of pointlike instantons (5-branes) (and possibly truncations). These give tensor multiplets for the  $E_8 \times E_8$  string [20], and vector multiplets for the  $Spin(32)/\mathbb{Z}_2$  string [19]. Compactifying on an extra circle, the 2 theories become T-dual, as reflected in our oxidation chain.



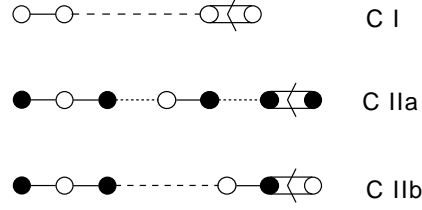


Figure 5: Satake diagrams for non-compact forms of the  $C_n$  groups

## 4.4 $C_n$

### 4.4.1 $C I : Sp(n+1)$

This is the split real form, analyzed before in [2] and [4].

Maximal compact subgroup, character and  $\mathbb{R}$  rank are given by:

$$H = U(n); \quad \sigma = n; \quad r = n$$

As this is a split form, obviously

$$G_s \times G_c = Sp(n, \mathbb{R}) \times \{e\}$$

These theories always oxidize to 4 dimensions; see [2] for details.

### 4.4.2 $C II: Sp(n-p, p)$

As  $Sp(a, b) \cong Sp(b, a)$ , we restrict to  $p \leq \frac{1}{2}n$ . In the list of Satake diagrams, we have furthermore distinguished between  $C IIa$ , for  $p \neq \frac{1}{2}n$ , and  $C IIb$  for  $p = \frac{1}{2}n$ .

Maximal compact subgroup, character and  $\mathbb{R}$ -rank are given by

$$H = Sp(p) \times Sp(n-p); \quad \sigma = -n - 2(n-2p)^2 \quad r = p$$

In all cases  $G_s \times G_c$  can be computed to be

$$G_s \times G_c = (SL(2, \mathbb{R})_2 \times SU(2)_2)^p \times Sp(n-2p)$$

We have added the subscript 2 to the  $SL(2, \mathbb{R})_2$  and  $SU(2)_2$  subgroups, to indicate that these are index 2 subgroups; they have short roots. There is no  $SL(n, \mathbb{R})$  subgroup with index 1, i.e. with long roots, and hence these theories do not oxidize.

The 3-d cosets with  $p = 1$  are quaternionic spaces [12].

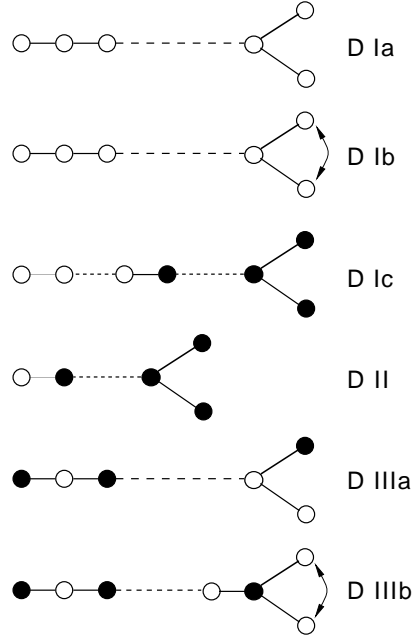


Figure 6: Satake diagrams for non-compact forms of the  $D_n$  groups

## 4.5 $D_n$

### 4.5.1 $D I, D II$ : $SO(2n-p, p)$

These models are similar to those based on  $B I, B II$  (subsection 4.3.1), and are often treated simultaneously in the literature.

Again, obviously  $SO(a, b) \cong SO(b, a)$ , and we restrict to  $p \leq n$ . The list of Satake diagrams distinguishes between  $D Ia$ , where  $p = n$  (the split form),  $D Ib$  where  $p = n - 1$ , and  $D Ic$ , with  $1 < p < n - 1$ . Finally  $D II$  refers to  $p = 1$ .

The maximal compact subgroup, character and  $\mathbb{R}$ -rank are given by

$$H = SO(2n-p) \times SO(p); \quad \sigma = n - 2(n-p)^2; \quad r = p$$

In all cases we have

$$G_s \times G_c = SO(p, p) \times SO(2n-2p)$$

The analysis of these theories is very similar to the one for the  $B I$  and  $B II$  theories. The

table of groups encountered in group disintegration is

$D$	$G/H$
$p+2$	$SO(2n-2p) \times \mathbb{R} / SO(2n-2p)$
$\vdots$	$\vdots$
$d+2$	$SO(2n-p-d, p-d) \times \mathbb{R} / SO(2n-p-d) \times SO(p-d)$
$\vdots$	$\vdots$
6	$SO(2n-p-4, p-4) \times \mathbb{R} / SO(2n-p-4) \times SO(p-4)$
5	$SO(2n-p-3, p-3) \times \mathbb{R} / SO(2n-p-3) \times SO(p-3)$
4	$SO(2n-p-2, p-2) \times SL(2, \mathbb{R}) / SO(2n-p-2) \times SO(p-2) \times SO(2)$
3	$SO(2n-p, p) / SO(2n-p) \times SO(p)$

Furthermore, there is an extra possibility if  $p > 3$ :

$D$	$G/H$
6	$SO(2n-p-3, p-3) / SO(2n-p-3) \times SO(p-3)$

Finally, for  $n = p = 1$  we have  $SO(1, 1) \cong \mathbb{R}$ , which does not lead to an oxidizable theory. For  $SO(2q+1, 1)$ , oxidation is never possible (note that  $SO(3, 1) \cong SL(2, \mathbb{C})$  and  $SO(5, 1) \cong SU^*(4)$ ).

In the other cases the theory can be oxidized to  $p+2$  dimensions, where one finds an Einstein-dilaton type gravity, coupled to an antisymmetric tensor, and  $2n-2p$  vectors. The latter form the vector representation of  $SO(2n-2p)$ , though this compact factor cancels from the sigma model.

There is (for  $p > 3$ ) a separate branch in 6 dimensions, leading to a theory with general relativity, a sigma model, and (anti-)self dual tensors. The number of self dual and anti-self dual tensors is only equal for  $p = n$ , hence for split forms. For the non-split real forms, the 6 dimensional theory is chiral.

As in the  $BI$  and  $BII$  cases, in  $p+2$  dimensions the Bianchi identity for the 2-form takes the form

$$dF_{(3)} = \frac{1}{2} \sum_{i=1}^{2n-2p} F_{(2)i} \wedge F_{(2)i} \quad (16)$$

again to be compared with the identity  $dH = \frac{1}{2}\text{Tr}(F \wedge F) - \frac{1}{2}\text{Tr}(R \wedge R)$  in string theory (but do note the possibility that  $p = n$ ). Also these models follow from the general considerations of [16] for string theories. For theories with 16 supersymmetries the models based on  $SO(8, 2q)$  are important, giving type I supergravity coupled to an even number of Yang-Mills multiplets in 10 dimensions [17].

The theories based on based on  $SO(4, 2q)$  (note that  $SO(4, 2) \cong SU(2, 2)$ ) give rise to quaternionic manifolds in 3 dimensions, and are important in the context of theories with 8 supersymmetries [13]. As in the  $B$  chains, a string theory realization can be constructed

by compactifying a heterotic string theory on  $K3$ , with pointlike instantons (5-branes). These give tensor multiplets for the  $E_8 \times E_8$  string [20], and vector multiplets for the  $Spin(32)/\mathbb{Z}_2$  string [19]. These theories are T-dual after compactification on an extra circle, as reflected in our oxidation chain.

#### 4.5.2 $D III$ : $SO^*(2n)$

This non-compact form has maximal compact subgroup and character

$$H = U(n); \quad \sigma = -n.$$

There is an important difference between  $n$  even or odd. If  $n$  is even, the Satake diagram is of type  $D IIIa$  and the  $\mathbb{R}$ -rank  $r = n/2$ . In this case, the groups  $G_s \times G_c$  are given by:

$$G_s \times G_c = SL(2)^{n/2} \times SU(2)^{n/2}.$$

If  $n$  is odd, the Satake diagram is of type  $D IIIb$  and the  $\mathbb{R}$ -rank  $r = (n-1)/2$ . Then the groups  $G_s \times G_c$  are given by

$$G_s \times G_c = SL(2)^{\frac{1}{2}(n-1)} \times SU(2)^{\frac{1}{2}(n-1)} \times U(1)$$

Irrespective of whether  $n$  is even or odd, the table of groups appearing in oxidation is:

$D$	$G/H$
4	$SO^*(2n-4) \times SU(2)/U(n-2) \times SU(2)$
3	$SO^*(2n)/U(n)$

The maximal dimension is always 4. The 4 dimensional theory has a graviton, and a sigma model on  $SO^*(2n-4)/U(n-2)$ . There are  $2n-4$  vectors, organizing with their duals in vectors of  $SO^*(2n-4)$ , and a doublet of the (hidden)  $SU(2)$  factor. A special case is  $SO^*(4) \cong SL(2, \mathbb{R}) \times SU(2)$ , but the previous statements still hold for these theories.

## 4.6 $E_6$

### 4.6.1 $E I$ : $E_{6(6)}$

This is the split form. Its maximal compact subgroup, character and  $\mathbb{R}$ -rank are:

$$H = Sp(4); \quad \sigma = 6 \quad r = 6.$$

As obvious for split form, the group  $G_c$  is trivial:

$$G_s \times G_c = E_{6(6)} \times \{e\}$$

The analysis of [2] (see also [4]) revealed that this theory can be oxidized to 8 dimensions.

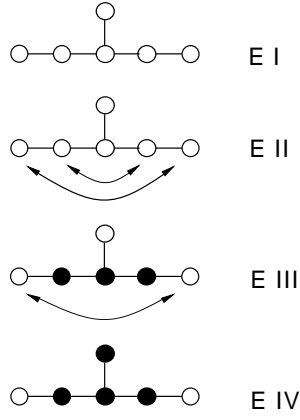


Figure 7: Satake diagrams for non-compact forms of the  $E_6$

#### 4.6.2 $E II$ : $E_{6(2)}$

This real form has maximal compact subgroup, character and  $\mathbb{R}$ -rank given by

$$H = SU(6) \times SU(2); \quad \sigma = 2; \quad r = 4.$$

The groups  $G_s$  and  $G_c$  are given by

$$G_s \times G_c = SL(4, \mathbb{R}) \times \mathbb{R} \times U(1)^2$$

The chain of groups appearing in the oxidation is

$D$	$G/H$
6	$SL(2, \mathbb{C}) \times U(1)/SU(2) \times U(1)$
5	$SL(3, \mathbb{C})/SU(3)$
4	$SU(3, 3)/S(U(3) \times U(3))$
3	$E_{6(2)}/SU(6) \times SU(2)$

The 6 dimensional theory consists of general relativity, a sigma model on  $SL(2, \mathbb{C})/SU(2)$ , 4 vectors, and 2 2-tensors. Actually this theory is a close relative of the 6 dimensional theory that can be oxidized from  $F_{4(4)}/(Sp(3) \times SU(2))$  [2]. The group  $E_6$  allows an outer automorphism, that is manifest in its Dynkin diagram. Decorating the Dynkin diagram with its outer automorphism, one precisely finds the Satake diagram  $E II$ . Quotienting  $E_6$  by its outer automorphism, one obtains  $F_4$ , and a computation reveals that the real form of  $F_4$  embedded in  $E_{6(2)}$  is  $F_{4(4)}$ . As pointed out in [2], the 6 dimensional theory oxidized from  $F_{4(4)}$  is an extension of a class of theories studied by Sagnotti [21]. We expect that the present 6 dimensional theory, oxidized from  $E_{6(2)}$ , allows a similar treatment.

The lower dimensional theories in the oxidation chain can also be found in [13].

#### 4.6.3 *E III: $E_{6(-14)}$*

This non-compact form has maximal compact subgroup, character and  $\mathbb{R}$ -rank

$$H = SO(10) \times U(1); \quad \sigma = -14; \quad r = 2$$

The groups  $G_s$  and  $G_c$  can be computed to be

$$G_s \times G_c = SL(2)^2 \times SU(4)$$

It oxidizes to 4 dimensions. The relevant groups are given by

$D$	$G/H$
4	$SU(5, 1)/U(5)$
3	$E_{6(-14)}/SO(10) \times U(1)$

This theory allows a supersymmetric extension: it corresponds to the bosonic sector of  $N = 5$  supergravity. The 4 dimensional model has a graviton, a sigma model on  $SU(5, 1)/U(5)$ , and 10 vectors, that, together with their duals, form the 20 dimensional antisymmetric 3-tensor irrep of  $SU(5, 1)$ .

#### 4.6.4 *E IV: $E_{6(-26)}$*

This non-compact form has maximal compact subgroup, character and  $\mathbb{R}$ -rank given by

$$H = F_4; \quad \sigma = -26; \quad r = 2.$$

Computation of the groups  $G_s$  and  $G_c$  leads to

$$G_s \times G_c = \mathbb{R}^2 \times SO(8)$$

As  $G_s$  has no  $SL(n, \mathbb{R})$  subgroup, this theory cannot be oxidized.

### 4.7 $E_7$

#### 4.7.1 *E V: $E_{7(7)}$*

This is the split form, that was analyzed in [2]. Maximal compact subgroup, character and  $\mathbb{R}$ -rank are given by

$$H = SU(8); \quad \sigma = 7; \quad r = 7$$

The groups  $G_s$  and  $G_c$  are

$$G_s \times G_c = E_{7(7)} \times \{e\}$$

as obvious for a split form

This theory oxidizes to 10 dimensions, but has an additional branch in 8 dimensions. For details we refer the reader to [2, 4].

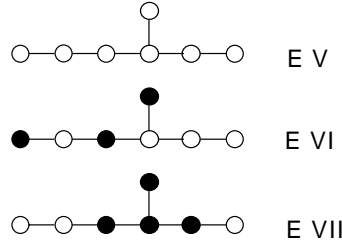


Figure 8: Satake diagrams for non-compact forms of the  $E_7$  groups

#### 4.7.2 $E_{VI}$ : $E_{7(-5)}$

This non-compact real form has maximal compact subgroup, character and  $\mathbb{R}$ -rank given by

$$H = SO(12) \times SU(2); \quad \sigma = -5; \quad r = 4$$

The groups  $G_s$  and  $G_c$  are given by

$$G_s \times G_c = SO(4, 4) \times SU(2)^3$$

We immediately see that this theory oxidizes to 6 dimensions. In the oxidation chain one finds the  $U$ -duality groups:

$D$	$G/H$
6	$SU(2) \times SU^*(4)/SU(2) \times Sp(2)$
5	$SU^*(6)/Sp(3)$
4	$SO^*(12)/U(6)$
3	$E_{7(-5)}/SO(12) \times SU(2)$

In 6 dimensions the theory oxidizes to the bosonic sector of a chiral  $(2, 1)$  supergravity, as conjectured in [7, 8] and demonstrated in [15]. The amount of supersymmetry in 6 dimensions is  $(2, 1)$ , and the chiral nature of the theory complicates its analysis. Its defining equations appear to have not been written down. With our present formalism it is straightforward to reconstruct the equations for the bosonic sector of the 6-d theory.

The same bosonic sector can be built from an  $N = 2$  theory (lower dimensional theories in the oxidation chain were discussed in [13]). This theory is closely related to the  $E_{6(2)}$  and  $F_{4(4)}$  theories; a projection of the  $E_{7(-5)}$  root lattice on the space  $\mathcal{H}_-$  defined by the Cartan involution gives the root lattice of  $F_{4(4)}$ , e.g. all these 3 theories have the same restricted root system with different multiplicities for the roots [14].

#### 4.7.3 $E_{VII}$ : $E_{7(-25)}$

This real form has maximal compact subgroup, character and  $\mathbb{R}$ -rank given by

$$H = E_6 \times U(1); \quad \sigma = -25; \quad r = 3.$$

A computation reveals that  $G_s$  and  $G_c$  are given by

$$G_s \times G_c = SL(2)^3 \times SO(8)$$

We immediately see that the theory can be oxidized to 4 dimensions. The relevant U-duality groups are

$D$	$G/H$
4	$SO(10, 2)/SO(10) \times SO(2)$
3	$E_{7(-25)}/E_6 \times U(1)$

In 4 dimensions, we find general relativity, coupled to 20 scalars in the coset, and 16 vectors that together with their duals transform in the 32 dimensional spinor irrep of  $SO(10, 2)$ .

## 4.8 $E_8$

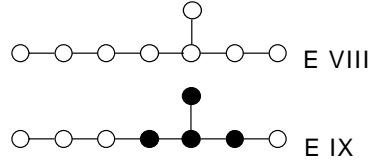


Figure 9: Satake diagrams for non-compact forms of the  $E_8$  groups

### 4.8.1 $E_{VIII}$ : $E_{8(8)}$

This famous coset oxidizes to the bosonic sector of 11 dimensional supergravity [22], and has a separate branch in 10 dimensions, where it corresponds to the bosonic sector of IIB gravity [23].

Its maximal compact subgroup, character and  $\mathbb{R}$ -rank, are given by

$$H = SO(16); \quad \sigma = 8; \quad r = 8$$

As the group is split, we obviously have

$$G_s \times G_c = E_{8(8)} \times \{e\}$$

The literature on theories in this chain is immense. Important original references are [22, 23, 24, 25], while a review is [26].



### 4.8.2 *E IX*: $E_{8(-24)}$

The algebra  $E_8$  has a second non-compact form, with maximal compact subgroup, character and  $\mathbb{R}$ -rank given by

$$H = E_7 \times SU(2); \quad \sigma = 6; \quad r = 6$$

The groups  $G_s \times G_c$  are given by

$$G_s \times G_c = SO(4, 4) \times SO(8)$$

This theory oxidizes to 6 dimensions, and the U-duality groups are given by:

$D$	$G/H$
6	$SO(9, 1)/SO(9)$
5	$E_{6(-26)}/F_4$
4	$E_{7(-25)}/E_6 \times U(1)$
3	$E_{8(-24)}/E_7 \times SU(2)$

In 6 dimensions, this theory includes general relativity, 9 scalars on  $SO(9, 1)/SO(9)$ , 16 vectors transforming as a spinor of  $SO(9, 1)$ , and 5 2-tensors.

The theories in this chain are well known in the context of the  $\mathbf{r}$  and  $\mathbf{c}$  maps [13]. The group  $E_{8(-24)}$  is again closely related to the groups  $E_{7(-5)}$ ,  $E_{6(2)}$  and  $F_{4(4)}$ ; all have the restricted root system (the roots projected on the invariant subspace  $\mathcal{H}_-$ ) of  $F_{4(4)}$ , with different multiplicities for the roots. All allow supersymmetric extensions.

Another amusing observation is that the 3 dimensional theories based on cosets formed by dividing  $F_{4(4)}$ ,  $E_{6(2)}$ ,  $E_{7(-5)}$  and  $E_{8(-24)}$  by there maximal compact subgroups, all oxidize to 6 dimensions, with sigma models on  $SL(2, \mathbb{R}) \cong SO(2, 1)$ ,  $SL(2, \mathbb{C}) \cong SO(3, 1)$ ,  $SL(2, \mathbb{H}) \cong SO(5, 1)$  and  $SL(2, \mathbb{O}) \cong SO(9, 1)$  respectively (see e.g. [27] for definitions of  $SL(2, \mathbb{H})$  and  $SL(2, \mathbb{O})$ ). These are all Lorentz groups; in the supersymmetric extensions of these theories this is dictated by  $D = 6$  (1,0) supersymmetry. The 2-tensors plus their duals transform in the vector representation of the Lorentz groups (also dictated by supersymmetry), while the vectors transform in a spinor representation. A lot of mathematical structure in the *bosonic* sector of these theories is familiar from supersymmetric Yang-Mills theories. Together with the links to division algebra's and exceptional groups, these oxidation chains provide interesting and entertaining mathematics.

## 4.9 $F_4$

### 4.9.1 *FI*: $F_{4(4)}$

This split form has maximal compact subgroup, character and  $\mathbb{R}$ -rank

$$H = Sp(3) \times SU(2); \quad \sigma = 4; \quad r = 4;$$

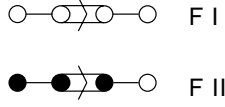


Figure 10: Satake diagrams for non-compact forms of  $F_4$

and

$$G_s \times G_c = F_{4(4)} \times \{e\}.$$

It oxidizes to 6 dimensions. An extensive discussion of the 6 dimensional theory is found in [2]. Lower dimensional theories from this chain can be found in [13].

#### 4.9.2 $F II$ : $F_{4(-20)}$

This non-compact form has maximal compact subgroup, character and  $\mathbb{R}$ -rank

$$H = SO(9); \quad \sigma = -20; \quad r = 1$$

The groups  $G_s \times G_c$  are computed to be

$$G_s \times G_c = SL(2, \mathbb{R})_2 \times Sp(3).$$

The subscript 2 on  $SL(2, \mathbb{R})$  denotes that we are dealing with an index 2 subgroup here. From this we immediately deduce that this theory cannot be oxidized, as was already known since [1].

### 4.10 $G_2$

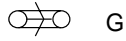


Figure 11: Satake diagrams for non-compact forms of  $G_2$

#### 4.10.1 $G$ : $G_{2(2)}$

The only non-compact form of the group  $G_2$  is the split form, with maximal compact subgroup, character and  $\mathbb{R}$ -rank

$$H = SO(4); \quad \sigma = 2; \quad r = 2$$

The groups  $G_s$  and  $G_c$  are obviously given by

$$G_s \times G_c = G_{2(2)} \times \{e\}$$

This theory oxidizes to the bosonic sector of simple supergravity in 5 dimensions. For an extensive study of the theories in this chain see [28].

## 5 Summary and remarks

We have extended the analysis of [4] to cover oxidation from all coset theories formulated on  $G/H$ , with  $G$  a non-compact simple Lie group, and  $H$  its maximal compact subgroup.

For these we had to deal with the Cartan involution. In the previous paper we had been able to ignore the Cartan involution, because for *all* split groups the Cartan involution can be chosen such that it acts as  $-1$  on the Cartan sub-algebra. As demonstrated in this paper, dealing with generic Cartan involutions hardly poses any problems; using some technology from group theory, we hardly need additional ingredients for the sigma model analysis.

An important ingredient in our analysis were the compact subgroups  $G_c$ . In particular, these were vital to our analysis of coset sigma models, that forms the basis of our oxidation recipe. They can be computed from the Satake diagram of the non-compact real form. Satake diagrams are also helpful in reading of the process of group disintegration; this basically follows the same pattern as for the split groups, where the analysis is done in terms of Dynkin diagrams. The extra decoration that accompanies the Satake diagram encodes the non-compact forms we find in the various dimensions

Together with the analysis in [4], the results of the present paper represent an exhaustive analysis. To some extent, because many of the theories discussed here were known before, our main contribution is the demonstration that we have exhausted all possibilities for oxidation from simple Lie groups. We remind the reader that this does however rely on the assertion that the higher dimensional theories follow from the possible index 1  $SL(D-2, \mathbb{R})$  subgroups [4]. We have explained that this is equivalent to demanding a theory with exactly 1 graviton, and other irreps allowing interpretations as form fields, and scalars. As there exist no-go theorems on theories with multiple gravitons, and massless fields that are not forms, this seems a reasonable requirement.

An elegant result is that the full bosonic sector of a large class of theories, among which many supergravity theories is encoded in a surprisingly small set of ingredients: essentially a Satake diagram (which compactly encodes the algebra, the roots, the Cartan involution, the relevant subalgebra's and possible physical dualities) and a set of equations: the dilaton equation (11), a single equation relating the form fields and axions (13), algebraic equations (8) for some axions and dilatons, and the Einstein equation, which states that the Einstein tensor couples to the energy momentum tensors of a set matter fields (which matter fields follows again from group theory).

A perhaps interesting observation is that in the formalism we used, the theories based on non-split groups can always be recovered from the split cases. We start with the split group, and build the theories with the recipe of [4]. Because of the choice of (positive root) gauge, a theory based on a non-split form can immediately be recovered by setting some fields to zero (see equation (8)). This is possible because the truncation of the model

based on the split group is equivalent to fixing the gauge in the other model; essentially we are turning some non-compact generators into compact ones, and then gauge them away. As it is the group  $G_c$  determining the set of fields in question, this once more emphasizes the important role this subgroup plays.

The wide range of applicability of the methods developed in [4] and this paper leads us to expect that they might be useful to other problems in the context of general relativity and supergravity. We hope to report on other applications in the future.

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